Exercises for “Vertiefung Neuronale Netze”
SS 2018 Sheet 3

Due on: 22.6.2018

Task 3.1, : Consider a 2-dimensional input space $\vec{x} = [x_1, x_2]^t$ with the following polynomial kernel:

$$k(\vec{x}, \vec{z}) \equiv \vec{\phi}(\vec{x}) \cdot \vec{\phi}(\vec{z}) = (\vec{x} \cdot \vec{z} + 1)^2 \tag{1}$$

Show that the associated feature function $\phi(x)$ is given by:

$$\vec{\phi}(\vec{x}) = [x_1^2, x_2^2, \sqrt{2}x_1 x_2, \sqrt{2}x_1, \sqrt{2}x_2, 1]^t \tag{2}$$

This means, the feature vector contains all terms up to the second order.

Task 3.2, : Show that the Gaussian kernel

$$k(x, z) = exp(-\|x - z\|^2) \tag{3}$$

is equivalent to a infinite dimensional feature vector. To do this prove the following:

- $\|x - z\|^2 = x^t \cdot x - 2x^t \cdot z + z^t \cdot z$
- $exp(-\|x - z\|^2) = exp(-\|x\|^2) \cdot exp(+2x^t \cdot z) \cdot exp(-\|z\|^2)$
- Given a kernel $k(x, z) = \phi(x) \cdot \phi(z)$ and $f(x)$ an arbitrary scalar function. In this case also $f(x) \cdot k(x, z) \cdot f(z)$ is a kernel, meaning that it can be represented by a scalar product
- By developing $exp(2x^t \cdot z)$into the infinite series, it can be represented as a scalar product in an infinite feature space. How does it look like? Use the Multinomial theorem:

$$(x_1 + \cdots + x_r)^n = \sum_{k_1 + \cdots + k_r = n} \binom{n}{k_1, \cdots, k_r} \cdot x_1^{k_1} \cdots x_r^{k_r} \tag{4}$$

$$\binom{n}{k_1, \cdots, k_r} = \frac{n!}{k_1! \cdots k_r!} \tag{5}$$